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Weighted Stochastic Sobolev Spaces and Bilinear SPDEs Driven by Space–Time White Noise

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In this paper we develop basic elements of Malliavin calculus on a weighted $L^2(\Omega)$. This class of generalized Wiener functionals is a Hilbert space. It turns out to be substantially smaller than the space of Hida distributions while large enough to accommodate solutions of bilinear stochastic PDEs. As an example, we consider a stochastic advection-diffusion equation driven by space-time white noise in $\mathbb{R}^d$. It is known that for $d > 1$, this equation has no solutions in $L^2(\Omega)$. In contrast, it is shown in the paper that in an appropriately weighted $L^2(\Omega)$ there is a unique solution to the stochastic advection-diffusion equation for any $d \geqslant 1$. In addition we present explicit formulas for the Hermite–Fourier coefficients in the Wiener chaos expansion of the solution.

1. INTRODUCTION

Consider a stochastic partial differential equation of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathscr{L}_+ u \cdot \dot{W} \\ u(0, x) = u_0(x), \end{cases} \tag{1.1}$$

where W is a white noise on $[0, T] \times \mathbb{R}^d$, and $\mathscr{L}$ is a uniformly elliptic second order differential operator.

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In physical terminology Eq. (1.1) is an advection-diffusion equation with random potential. Not surprisingly this equation turns out to be a convenient limit model for various phenomena in physics, biology and other sciences. In this capacity it has been studied intensively by many authors (see e.g. Carmona, Molchanov [1], Kifer [7], Mueller [15], Zakai [22], etc.)

Properties of Eq. (1.1) depend crucially on the type of the Brownian motion $W(t) = \int_0^t \dot{W}(ds)$. It "behaves" rather nicely if the space covariance operator Q defined by $E(\langle W(t), f\rangle \langle W(t), g\rangle) = \langle Qf, g\rangle\, t$ is nuclear. In this case there exists a well developed theory of bilinear SPDEs (see e.g. Rozovskii [20], DaPrato, Zabczyk [2], Krylov [9], Mikulevicius, Rozovskii [14], and references therein).

On the contrary if $\dot{W}$ is a space-time white noise ($Q \equiv I$), the theory is rather patchy and incomplete.

In this case Eq. (1.1) is well studied only if the dimension $d = 1$ (see [9, 15]) and references therein). For $d > 1$ it turns out that Eq. (1.1) has no square integrable (in probability) solution in any Sobolev space with positive or negative index (see Remark 8 in Section 3).

In [18] Nualart and Zakai proved that the solution $u(t, x)$ to this equation in dimension $d > 1$ is given by a formal series of multiple stochastic integrals

$$u(t, x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)),$$

where the kernels $f_n(\cdot, t, x)$ do not belong to $L^2(([0, T] \times \mathbb{R}^d)^n)$. Actually they are in $L^p(([0, T] \times \mathbb{R}^d)^n)$, for $p < d/(d-1)$.

Unfortunately, a solution to Eq. (1.1) in the sense of Nualart–Zakai is not a collection of random variables parameterized by the space, time and chance parameters (x, t, ω) and not even by smooth test functions of space and time.

Later on this equation was studied by Holden, Øksendal, Ubøe, and Zhang (see [6] and the references therein), and Potthoff, Våge and Watanabe [19]. In the former paper existence of a solution was proved in a class of white noise functional processes (i.e., a family of random variables parameterized by smooth test functions). Uniqueness of the solution and a generalized Feynman–Kac formula were also established in this paper. In the latter paper similar results were obtained in the Hida space $(\mathscr{S})^*$ of generalized distributions.

In the present paper we show that in fact the "stochastic support" of a solution to Eq. (1.1) can be characterized more precisely. Specifically, we demonstrate that it belongs to an appropriately weighted Gaussian $L^2(\Omega)$.

The structure of this space is quite simple. Given a positive self-adjoint operator Q on $L^2(\mathbb{R}^d)$, we define $L^2_Q(\Omega)$ as a completion of the space of Wiener functional $F \in L^2(\Omega)$ in the norm

$$\|F\|_Q^2 := \sum_{\alpha \in J} (\rho^\alpha)^2 \, E(F\xi_\alpha)^2,$$

where J is the set of multiindices of finite length, $\{\xi_\alpha\}$ is the basis in Gaussian $L^2(\Omega)$ consisting of Wick polynomials associated with $\dot{W}$, and $\rho^\alpha := \prod_{\alpha_i \in \alpha} \rho_i^{\alpha_i}$ where ρ_i are the eigenvalues of $Q^{1/2}$.

Note that the space of distributions $L^2_{Q^-}(\Omega) := (L^2_Q(\Omega))^*$ coincides with the space $\mathbb{D}^1_{2,0}$ in the notation of Körezlioglu and Üstünel [8] (see also Kubo, Yokoi [10], and Meyer, Yan [12]). In a special case when the operator

$$Q = \left(\prod_{i=1}^d \left(\frac{\partial^2}{\partial x_i^2} + x_i^2 + 1 \right) \right)^{-2},$$

$L^2_Q(\Omega)$ coincides with one of the Hilbert subspaces of the Hida space $(\mathscr{S})^*$ (for details see Section 2).

The paper consists of an Introduction and two sections. The basic elements of stochastic analysis on $L^2_Q(\Omega)$ are developed in Section 2.

In Section 3 we show that there exists a unique solution $u(t, x)$ to Eq. (1.1) so that its norm in $L^2_Q(\Omega)$ is locally bounded in t, x. This result holds under minimal assumptions (Hölder continuity) on the coefficients of the operator $\mathscr{L}$, and provided that in dimensions $d > 1$, the operator Q is Hilbert–Schmidt and satisfies an integrability condition (see condition B_2 in Section 3). In particular this condition holds if the kernel $Q(x, y)$ of the operator Q verifies

$$\sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} Q(z, y)^2 \, dy < \infty.$$

If only Hölder continuity of the coefficients of the operator $\mathscr{L}$ and the initial value u_0 is assumed, we understand the solution to Eq. (1.1) as a soft solution, in the sense of [13, 14]. We demonstrate that if appropriate additional regularity of the coefficients of $\mathscr{L}$ is allowed, then the soft solution to (1.1) coincides with the standard generalized solution. In both cases we derive "explicit" formulas for the Hermite–Fourier coefficients of the Cameron–Martin development of the solution to Eq. (1.1), the Feynman–Kac formula and the maximum principle.

We remark that the results of the paper can be extended to a more general case of equation $\dot{u} = \mathscr{L}u + \mathscr{M}u \cdot \dot{W}$ where $\mathscr{M}u = \sum_{i=1}^d \sigma_i(t, x) \, u_{x_i} + h(t, x) \, u$. However, for the sake of brevity, we refrain from doing so in the present paper. This interesting problem will be addressed elsewhere.

2. WEIGHTED SOBOLEV SPACES

Let H be a real separable Hilbert space. Consider a zero-mean Gaussian family $W=\{W(h), h\in H\}$ defined on a complete probability space $(\Omega, \mathscr{F}, P)$ and such that $E(W(h)\,W(g))=\langle h, g\rangle_H$ for all $g, h\in H$. We will assume that the σ-field $\mathscr{F}$ is generated by W. The Gaussian process W will be called a white noise on the Hilbert space H.

Fix a complete orthonormal system $\{e_k, k\geqslant 1\}$ on H and consider a non negative self-adjoint operator Q on H such that $Q^{1/2}e_k=\rho_k e_k$, where $\{\rho_k, k\geqslant 1\}$ is a nonincreasing sequence of positive numbers such that $\rho_1\leqslant 1$.

We will make use of the following notations. $\mathscr{J}$ is the set of all multi-indices $\alpha=(\alpha_k, k\geqslant 1)$ such that $\alpha_k\in\mathbb{N}=\{0, 1, 2, 3, ...\}$ and $|\alpha| := \sum_{k=1}^{\infty}\alpha_k<\infty$. For any $\alpha\in\mathscr{J}$ we set $\alpha! = \prod_{k=1}^{\infty}\alpha_k!$, and $\rho^{\alpha}=\prod_{k=1}^{\infty}\rho_k^{\alpha_k}$. We will denote by H_n the nth Hermite polynomial defined by $H_n(x)=(-1)^n((d^n/dx^n)\,e^{-x^2/2})\,e^{x^2/2}$. Consider the family of random variables

$$\xi_\alpha := \prod_{k=1}^{\infty} H_{\alpha_k}(W(e_k))/\sqrt{\alpha!}, \qquad \alpha=(\alpha_k)\in\mathscr{J}.$$

We know that $\{\xi_\alpha, \alpha\in\mathscr{J}\}$ form a complete orthonormal system in $L^2(\Omega)$.

For any random variable $F\in L^2(\Omega)$ we define the norm

$$\|F\|_Q^2=\sum_{\alpha\in\mathscr{J}}(\rho^\alpha)^2\,E(F\xi_\alpha)^2.$$

Let us denote by $L_Q^2(\Omega)$ the completion of $L^2(\Omega)$ by the norm $\|\cdot\|_Q$. The elements of $L_Q^2(\Omega)$ will be called Q-distributions. The dual of $L_Q^2(\Omega)$ is the space

$$L_{Q^-}^2(\Omega)=\{F\in L^2(\Omega): \|F\|_{Q^-}<\infty\}, \tag{2.1}$$

where

$$\|F\|_{Q^-}^2=\sum_{\alpha\in\mathscr{J}}(\rho^\alpha)^{-2}\,E(F\xi_\alpha)^2.$$

Let us now introduce some elements of Malliavin Calculus. We will denote by $\mathscr{S}$ the class of random variables of the form

$$F=f(W(h_1), ..., W(h_n)), \tag{2.2}$$

where f belongs to $C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded), $h_1, ..., h_n$ are in H, and $n\geqslant 1$. These random variables are called *elementary*. The derivative operator D is defined on an elementary random variable F of the form (2.2) as the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (W(h_1), ..., W(h_n))\, h_i. \tag{2.3}$$

In particular we have

$$\begin{aligned} D\xi_\alpha &= \sum_{k=1}^{\infty} \left(\prod_{j \neq k} H_{\alpha_j}(W(e_j))/\sqrt{\alpha_j!} \right) \alpha_k H_{(\alpha_k - 1)^+}(W(e_k))\, e_k/\sqrt{\alpha_k!} \\ &= \sum_{k=1}^{\infty} \sqrt{\alpha_k}\, \xi_{\alpha^{k-}}\, e_k, \end{aligned}$$

where

$$\alpha_j^{k-} = \begin{cases} \alpha_j, & \text{if } j \neq k \\ (\alpha_k - 1)^+, & \text{if } j = k. \end{cases}$$

Notice that $E(\|D\xi_\alpha\|_H^2) = |\alpha|$. The domain of the derivative operator D in L^2 is

$$\mathbb{D}^{1,2} = \left\{ F \in L^2(\Omega) : \sum_{\alpha \in \mathcal{J}} |\alpha|\, E(F\xi_\alpha)^2 < \infty \right\}.$$

The operator D is continuous from $\mathbb{D}^{1,2}$ into $L^2(\Omega; H)$. Let us denote by δ the adjoint of the operator D. That is, the following duality relation holds for any $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom} \delta$:

$$E(F\delta(u)) = E(\langle DF, u \rangle_H). \tag{2.4}$$

The operator δ can be computed in terms of the expansion of a vector-valued random variable $u \in L^2(\Omega; H)$ into the basis $\{\xi_\alpha \otimes e_k, \alpha \in \mathcal{J}, k \geqslant 1\}$. We have

$$\delta(\xi_\alpha \otimes e_k) = \xi_\alpha W(e_k) - \langle D\xi_\alpha, e_k \rangle_H = \xi_\alpha W(e_k) - \xi_{\alpha^{k-}} \sqrt{\alpha_k}.$$

Using the properties of Hermite polynomials we can write

$$\xi_\alpha W(e_k) = \xi_{\alpha^{k+}} \sqrt{\alpha_k + 1} + \xi_{\alpha^{k-}} \sqrt{\alpha_k},$$

where

$$\alpha_j^{k+} = \begin{cases} \alpha_j, & \text{if } j \neq k \\ \alpha_k + 1, & \text{if } j = k. \end{cases}$$

Hence,

$$\delta(\xi_\alpha \otimes e_k) = \xi_{\alpha^{k+}} \sqrt{\alpha_k + 1}.$$

Obviously for $u \in L^2(\Omega; H)$ we can write

$$u = \sum_{\alpha \in \mathcal{J}} \sum_{k=1}^{\infty} E(\langle u, e_k \rangle_H \xi_\alpha)\, \xi_\alpha \otimes e_k.$$

Then, if $u \in \operatorname{Dom} \delta$ we have

$$\delta(u) = \sum_{\alpha \in \mathcal{J}} \sum_{k=1}^{\infty} E(\langle u, e_k \rangle_H \xi_\alpha)\, \xi_{\alpha^{k+}} \sqrt{\alpha_k + 1}.$$

The space $\mathbb{L}^{1,2} := \mathbb{D}^{1,2} \otimes H$ is included into the domain of δ, and for any $u \in \mathbb{L}^{1,2}$ we have

$$\begin{aligned} E(\delta(u)^2) &= E(\|u\|_H^2) + E(\|(Du)(Du)^*\|_{H \otimes H}) \\ &\leqslant E(\|u\|_H^2) + E(\|D\|_{H \otimes H}^2). \end{aligned} \tag{2.5}$$

Now using the operator Q we will introduce an extension of the operator $\delta(u)$. For this purpose we first restrict the domain of the derivative operator D to the space

$$\mathbb{D}_{Q^-}^{1,2} = \left\{ F \in L_0^2(\Omega) : \sum_{\alpha \in \mathcal{J}} |\alpha| (\rho^\alpha)^{-2} E(F\xi_\alpha)^2 < \infty \right\}.$$

Let us define

$$\operatorname{Dom}_Q \delta = \{ u \in L_Q^2(\Omega; H) : |E(\langle DF, u \rangle_H)| \leqslant c_u \|F\|_{Q^-}, \forall F \in \mathbb{D}_{Q^-}^{1,2} \}.$$

Then, for any $u \in \operatorname{Dom}_Q \delta$, $\delta(u)$ will be the unique element in $L_Q^2(\Omega)$ such that the duality relation

$$E(F\delta(u)) = E(\langle DF, u \rangle_H),$$

holds for any $F \in \mathbb{D}_{Q^-}^{1,2}$. It is readily checked that the space

$$\mathbb{D}_Q^{1,2} = \left\{ F \in L_Q^2(\Omega) : \sum_{\alpha \in \mathcal{J}} |\alpha| (\rho^\alpha)^2 E(F\xi_\alpha)^2 < \infty \right\}$$

can be identified with the dual to $\mathbb{D}_{Q^-}^{1,2}$ with respect to the duality defined by the scalar product in $L^2(\Omega)$. Moreover, the operator D is continuous from $\mathbb{D}_Q^{1,2}$ into $L_Q^2(\Omega; H)$. The following proposition is an immediate generalization of (2.5):

PROPOSITION 2.1. *The space* $\mathbb{L}_Q^{1,2} := \mathbb{D}_Q^{1,2} \otimes H$ *is contained into* $\operatorname{Dom}_Q \delta$, *and for all* $u \in \mathbb{L}_Q^{1,2}$ *we have*

$$E(\|\delta(u)\|_Q^2) = E(\|Q^{1/2}u\|_H^2) + E(\|(DQ^{Q/2}u)(DQ^{1/2}u)^*\|_{H\otimes H})$$
$$\leqslant E(\|Q^{1/2}u\|_H^2) + E(\|(DQ^{1/2}u)\|_{H\otimes H}^2).$$

Notice that the norm in $L^2_{Q^-}(\Omega)$ can be written as

$$\|F\|_{Q^-}^2 = \sum_{N=0}^{\infty} \sum_{\alpha\in\mathcal{J},\, |\alpha|=N} \left(\prod_{k=1}^{\infty} \rho_k^{-2\alpha_k}\right) E(F\xi_\alpha)^2$$
$$\leqslant \sum_{\alpha\in\mathcal{J}} \rho_1^{-2|\alpha|} E(F\xi_\alpha)^2.$$

On the other hand the elements F in the Sobolev space $\mathbb{D}^{\infty,2} := \bigcap_k \mathbb{D}^{k,2}$ are characterized by the property

$$\sum_{\alpha\in\mathcal{J}} |\alpha|^m\, E(F\xi_\alpha)^2 < \infty \qquad \text{for all} \quad m \in \mathbb{N}.$$

Hence, we get that $L^2_{Q^-}(\Omega) \subset \mathbb{D}^{\infty,2}$, and, by duality, $\mathbb{D}^{-\infty,2} \subset L^2_Q(\Omega)$.

Remark 1. Let A be a self-adjoint positive operator on H with domain $\operatorname{Dom}(A)$ and bounded inverse. Suppose in addition that $\inf \sigma(A) > 1$ and $H_\infty := \bigcap_n \operatorname{Dom}(A^n)$ is dense in H. We denote by $\Gamma(A)$ the second quantification of the operator A which is defined by

$$\Gamma(A)(e^{\delta(h) - 1/2\, \|h\|_H^2}) = e^{\delta(Ah) - 1/2\, \|Ah\|_H^2};$$

for any $h \in \operatorname{Dom}(A)$. For any integer p we introduce the following norm on a polynomial random variable $F \in \mathcal{P}$ ($\mathcal{P}$ denotes the family of random variables of the form (2.2) such that f is a polynomial and $h_i \in H_\infty$):

$$\|F\|_{2,p,A} = \|\Gamma(A^p)\, F\|_{L^2(\Omega)}.$$

Let us denote by $(L^2)_{p,A}$ the completion of the class $\mathcal{P}$ with respect to the norm $\|\cdot\|_{2,p,A}$. Set $(L^2)_{\infty,A} = \bigcap_p (L^2)_{p,A}$ and $(L^2)_{-\infty,A} = \bigcup_p (L^2)_{p,A}$. The space of test functions $(L^2)_{\infty,A}$ and the corresponding space of distributions $(L^2)_{-\infty,A}$ have been extensively used in the framework of white-noise analysis (see [5, 6], and the references therein). We recall that $(L^2)_{\infty,A}$ is an algebra and it is a nuclear provided for some $k \in \mathbb{N}$ the operator A^{-k} is Hilbert–Schmidt.

Observe that if we take $A = Q^{-1/2}$ then the space of test functions $L^2_{Q^-}(\Omega)$ defined in (2.1) coincides with $(L^2)_{1,Q^{-1/2}}$, and the associated space of distributions $L^2_Q(\Omega)$ is equal to $(L^2)_{-1,Q^{-1/2}}$.

The space of distributions $(\mathscr{S})^*$ introduced by Hida (see [5]) is a particular case of $(L^2)_{\infty, A}$ when $H = L^2(\mathbb{R}^d)$ and

$$A = \prod_{i=1}^{d} \left(-\frac{\partial^2}{\partial x_i^2} + x_i^2 + 1 \right).$$

In this case a sequence of orthonormal eigenvectors $\{e_k, k \geqslant 1\}$ of A is obtained by setting $e_k = \xi_{\gamma(k)_1} \otimes \cdots \otimes \xi_{\gamma(k)_d}$, where $\{\gamma(k), k \geqslant 1\}$ is a reordering of the set of d-dimensional multiindices $\{(k_1, ..., k_d), k_j \geqslant 0\}$, and for each $k \geqslant 0$, ξ_k denotes the Hermite function

$$\xi_k(x) = \pi^{-1/4} (k!)^{-1/2}\, 2^{-k/2} (-1)^{k+1}\, e^{x^2/2} \frac{d^k}{dx^k} (e^{-x^2}).$$

The corresponding eigenvalues are $\prod_{j=1}^{d} (2\gamma(k)_j + 2)$.

3. BILINEAR PARABOLIC EQUATIONS DRIVEN BY SPACE–TIMES WHITE NOISE

We will denote by H the Hilbert space $L^2([0, T] \times \mathbb{R}^d)$ and by W a white noise on H. The main objective of this section is to study the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathscr{L} u + u \cdot \dot{W} \\ u(0, x) = u_0(x), \end{cases} \tag{3.1}$$

where $\mathscr{L}$ is the second order differential operator given by

$$\mathscr{L} u = \frac{1}{2} \sum_{i, j=1}^{d} a^{ij}(t, x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{d} b^i(t, x) \frac{\partial u}{\partial x_i} + c(t, x)\, u.$$

We will assume that the coefficients

$$a\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d^2}, \qquad b\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \qquad c\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$$

are continuous functions. We will make use of the following assumptions on the coefficients of the operator $\mathscr{L}$.

Let $C^{2+\beta}([0, T] \times R^d)$, $\beta \in (0, 1)$ be the space of all continuous functions v on $[0, T] \times R^d$ with the finite norm

$$|v|_{2, \beta} = \sup_{t, x} |v(t, x)| + |\mathscr{D}_x^2 v|_\beta,$$

where

$$|v|_\beta = \sup_{t,\, x} |v(t, x)| + \sup_{t,\, x \neq y} \frac{|v(t, x) - v(t, y)|}{|x-y|^\beta}.$$

A_1: The matrix a^{ij} is uniformly elliptic, that is, for some $\delta > 0$ we have

$$\sum_{i,\, j=1}^{d} a^{ij}(t, x)\, \xi_i \xi_j \geqslant \delta\, |\xi|^2,$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$.

A_2: For $\beta \in (0, 1)$, $|a|_\beta + |b|_\beta + |c|_\beta < \infty$.

A_3: $u_0 \in C^{2+\beta}(R^d)$.

In order to introduce a notion of solution for this equation we consider an operator Q on $L^2(\mathbb{R}^d)$ given by $Q^{1/2} e_k = \rho_k e_k$, where $\{e_k, k \geqslant\}$ is a CONS on $L^2(\mathbb{R}^d)$ and ρ_k is a decreasing sequence of positive real numbers such that $\rho_1 \leqslant 1$. If we fix a CONS $\{m_j, j \geqslant 1\}$ in $L^2([0, T])$ formed by bounded and smooth functions, the operator Q can be extended to the Hilbert space H by putting

$$Q^{1/2}(m_j \otimes e_k) = m_j \otimes Q^{1/2} e_k = \rho_k (m_j \otimes e_k).$$

If the operator Q is Hilbert–Schmidt (i.e., $\sum_{k=1}^\infty \rho_k^4 < \infty$) we will denote by $Q(x, y)$ its kernel (which belongs to $L^2(\mathbb{R}^d \times \mathbb{R}^d)$) defined by

$$Q(x, y) = \sum_{k=1}^{\infty} \rho_k^2 e_k(x)\, e_k(y).$$

We can now introduce the norms $\|\cdot\|_Q$ and $\|\cdot\|_{Q^-}$ and the associated Sobolev spaces as in Section 2. The only difference is that here $\mathcal{J}$ will be a set of double sequences of indices,

$$\mathcal{J} = \left\{ \alpha = (\alpha_{jk},\, j, k \geqslant 1),\, \alpha_{jk} \in \mathbb{N},\, |\alpha| := \sum_{j,\, k=1}^{\infty} \alpha_{jk} < \infty \right\},$$

and the associated weights are $\rho_k^{\alpha_{jk}}$. We will write $\rho^\alpha = \prod_{j,\, k=1}^\infty \rho_k^{\alpha_{jk}}$, and with these notations we proceed as in Section 2. Notice that the first index refers to the time while the second index refers to the space variable.

We will denote by $\mathcal{F}_t$, $t \in [0, T]$, the σ-field generated by the random variables $W(1_{[0, s] \times B})$, where $0 \leqslant s \leqslant t$ and B is a bounded Borel subset of

$\mathbb{R}^d$. We will denote by $L^2_Q(\Omega, \mathscr{F}_t)$ the set of $\mathscr{F}_t$-measurable Q-distributions (or generalized random variables), which can be defined as

$$L^2_Q(\Omega, \mathscr{F}_t) = \{F \in L^2_Q(\Omega) : E(FG) = E(FE(G \mid \mathscr{F}_t)), \forall G \in L^2_{Q^-}(\Omega)\}.$$

We will denote by $L^{2,a}_Q(\Omega; H)$ the space of generalized random fields $u \in L^2_Q(\Omega; H) \cong L^2([0, T] \times \mathbb{R}^d, L^2_Q(\Omega))$ such that $u(t, x) \in L^2_Q(\Omega, \mathscr{F}_t)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$ a.e. and $\int_0^T \int_{\mathbb{R}^d} \|u(t, x)\|^2_Q \, dx \, dt < \infty$.

It is known that the space $L^{2,a}(\Omega; H)$ of square integrable processes $u \in L^2([0, T] \times \mathbb{R}^d; L^2(\Omega))$ which are adapted (i.e., $u(t, x)$ is $\mathscr{F}_t$-measurable for each (t, x)) is included into the domain of the operator δ and δ restricted to $L^{2,a}(\Omega; H)$ coincides with the Itô stochastic integral (see, for instance Nualart and Zakai [17]). The following proposition extends this property to generalized random fields.

Proposition 3.1. *The space of adapted generalized random fields $L^{2,a}_Q(\Omega; H)$ is contained in* $\mathrm{Dom}_Q\, \delta$.

Proof. Given an adapted generalized random field $u \in L^{2,a}_Q(\Omega; H)$ it can be approximated by elementary adapted processes. The rest of the proof follows as in [16, Proposition 1.3.4]. Q.E.D.

For any element $h \in H$ we will make use of the notation

$$q(h) = \exp\left(\delta(h) - \tfrac{1}{2} \int_0^T \int_{\mathbb{R}^d} h^2(t, x) \, dx \, dx\right).$$

It turns out that $q(h) \in \mathbb{D}^{1,2}_{Q^-}$. We will denote by H_0 the set of functions $h \in H \cap C^\beta([0, T] \times \mathbb{R}^d)$.

Suppose that $u = \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ is a generalized random field such that $u(t, x) \in L^2_Q(\Omega)$ for each (t, x). Then if $h \in H$ we define

$$u^h(t, x) = E(u(t, x)\, q(h)).$$

This definition coincides with the notion of S-transform used in white noise analysis (see [5]).

We can now introduce the notion of Q-soft solution for Eq. (3.1) following the ideas of Mikulevicius and Rozovskii (see [13] and [14]).

Definition 3.2. An adapted generalized random field $u = \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ such that $\int_0^T \int_{\{|x| \leqslant K\}} \|u(t, x)\|^2_Q \, dx \, dt < \infty$ for each $K > 0$, is said to be a Q-soft solution to Eq. (3.1) if for every function $h \in H_0$

the deterministic function $u^h(t, x)$ is a generalized solution to the parabolic equation

$$\begin{cases} \dfrac{\partial u^h}{\partial t} = \mathscr{L} u^h + u^h h \\ u^h(0, x) = u_0(x). \end{cases} \tag{3.2}$$

To motivate this definition let us assume for the moment that in addition to A_1–A_3 the following assumption holds.

A_4: The derivatives $\partial a^{ij}/\partial x_j$, $\partial^2 a^{ij}/\partial x_i \partial x_j$, and $\partial b^i/\partial x_i$ exist and are bounded functions.

Let us denote by $\mathscr{L}^*$ the adjoint of the operator $\mathscr{L}$ given by

$$\mathscr{L}^* u = \frac{1}{2} \sum_{i, j=1}^{d} a^{ij}(t, x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{d} (b^*)^i (t, x) \frac{\partial u}{\partial x_i} + (c^*)(t, x)\, u,$$

where

$$(b^*)^i = -b^i + \frac{1}{2} \sum_{j=1}^{d} \frac{\partial a^{ij}}{\partial x_j}$$

$$(c^*) = c - \sum_{i=1}^{d} \frac{\partial b^i}{\partial x_i} + \sum_{i, j=1}^{d} \frac{\partial^2 a^{ij}}{\partial x_i \, \partial x_j}.$$

Then, for each t in $[0, T]$ and for any test function $\varphi \in C_K^\infty(\mathbb{R}^d)$ we have

$$(u^h(t), \varphi)_0 = (u_0, \varphi) + \int_0^t (u^h(s), \mathscr{L}^* \varphi)_0 \, ds + \int_0^t \int_{\mathbb{R}^d} (u^h h)(s, x)\, \varphi(x)\, dx\, dx, \tag{3.3}$$

where $(\cdot, \cdot)_0$ denotes the scalar product in $L^2(\mathbb{R}^d)$.

Set $G = q(h)$, and notice that $D_{s, x} G = h(s, x)\, G$. Hence, Eq. (3.3) is equivalent to

$$E(G(u(t), \varphi)_0) = E(G(u_0, \varphi)_0) + \int_0^t E(G(u(s), \mathscr{L}^* \varphi)_0)\, ds + \int_0^t \int_{\mathbb{R}^d} E(u(s, x)\, \varphi(x)\, D_{s, x} G)\, dx\, ds. \tag{3.4}$$

Taking into account the definition of the operator δ as the adjoint of the derivative operator, Eq. (3.4) can be rewritten as

$$E(G(u(t),\varphi)_0)=E(G(u_0,\varphi)_0)+\int_0^t E(G(u(s),\mathscr{L}^*\varphi)_0)\,ds \\ +E(G\delta(\mathbf{1}_{[0,t]}u\varphi)). \tag{3.5}$$

Finally using the density of the random variables of the form $G=q(h)$, $h\in H_0$, in $L^2(\Omega)$, Eq. (3.5) is equivalent to

$$(u(t),\varphi)_0=(u_0,\varphi)_0+\int_0^t (u(s),\mathscr{L}^*\varphi)_0\,ds+\delta(\mathbf{1}_{[0,t]}u\varphi), \tag{3.6}$$

Note that the last summand in Eq. (3.6) is well defined because $\mathbf{1}_{[0,T]}u\varphi\in L_Q^{2,a}(\Omega;H)\subset \mathrm{Dom}_Q\,\delta$. Thus we have established the following result:

Proposition 3.3. *Suppose that hypotheses A_1 to A_4 hold. Consider an adapted generalized random field $u=\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ verifying the integrability condition $\int_0^T\int_{\{|x|\leqslant K\}}\|u(t,x)\|_Q^2\,dx\,dt<\infty$ for each $K>0$. Then u is a Q-soft solution to Eq. (3.1) if and only if for any $t\in[0,T]$ and for any test function $\varphi\in C_K^\infty(\mathbb{R}^d)$ Eq. (3.6) is satisfied.*

Before proceeding with the proof of existence of soft solutions we shall recall some facts regarding the deterministic equation (3.2).

It is a standard fact (see e.g., [3, 4]) that for any $h\in H_0$ under assumptions A_1–A_3 this equation has a unique solution in the Hölder space $C^{2+\beta}([0,T]\times R^d)$. Now we will construct a stochastic diffusion associated with Eq. (3.2) to discuss the Feynman–Kac representation of the solution to this equation.

Let $\mathscr{X}=C([0,T];\mathbb{R}^d)$ be the space of continuous $\mathbb{R}^d$-valued functions on $[0,T]$. As usual we define the canonical process on $\mathscr{X}$ by the formula $X_s=X_s(w)=w(s)$, $w\in\mathscr{X}$. Now we can equip $\mathscr{X}$ with the canonical σ-algebra $\mathscr{C}=\sigma(X_s,s\in[0,T])$ and the backward filtration $\mathscr{C}_s=\sigma(X_r, s\leqslant r\leqslant T)$. For any function $f(x)$ in the space $C_b^2(\mathbb{R}^d)$ of the twice continuously differentiable functions bounded with all their derivatives we define the operator $\mathscr{L}_s$ for $s\leqslant T$ by

$$\mathscr{L}_s f(x)=\frac{1}{2}\sum_{i,j=1}^d a^{ij}(s,x)\frac{\partial^2 f(x)}{\partial x_i\,\partial x_j}+\sum_{i=1}^d b^i(s,x)\frac{\partial f(x)}{\partial x_i}.$$

Definition 3.4. Given $(t,x)\in(0,T]\times\mathbb{R}^d$, a solution to the backward martingale problem for $\mathscr{L}_s$ starting from (t,x) is a probability measure $P^{t,x}$ on $(\mathscr{X},\mathscr{C})$ satisfying these properties:

(i) $P^{t,x}(X_s = x, t \leqslant s \leqslant T) = 1$;

(ii) $f(s, X_s) - f(t, x) + \int_s^t (f'(u, X_u) - \mathscr{L}_u f(u, X_u))\,du$ is a $P^{t,x}$ backward martingale (relative to the filtration $\mathscr{C}_s$) for all $f \in C_b^{1,2}([0, T] \times R^d)$.

The following proposition is a reformulation of the well-known result of Stroock and Varadhan (see e.g., [21]) in a way convenient for our purpose.

PROPOSITION 3.5. 1° *Assume* A_1 *and* A_2. *Then the backward martingale problem* (i), (ii) *has a unique solution.*

2° *If* A_1–A_3 *holds, then for any function* $h \in H_0$ *a solution of Eq.* (3.2) *is of the form*

$$u^h(t, x) = E^{t,x}\left(u_0(X_0) \exp\left(\int_0^t (c(x, X_s) + h(s, X_s))\,ds\right)\right), \tag{3.7}$$

where $E^{t,x}$ *stands for the integral with respect to* $P^{t,x}$.

3° *Assume* A_1, A_3. *Then the equation*

$$\frac{\partial v(s, x)}{\partial s} = \mathscr{L}_s v(s, x), \qquad (s, x) \in (0, T] \times \mathbb{R}^d$$

has a unique fundamental solution $P_{t,s}(x, y)$ *and given* $s \leqslant t$ *the following formula holds*

$$E^{t,x} f(X_s) = \int_{\mathbb{R}^d} f(y)\, p_{t,s}(x, y)\,dy, \qquad \textit{for all} \quad f \in L^\infty(\mathbb{R}^d).$$

Proof. Obviously the backward martingale problem can be reduced to the standard (forward) one by the change of variable $u = T - s$. So 1° follows from the classic uniqueness-existence result for martingale problems (see [21]).

To prove 2°, apply the Itô formula for the local square integrable martingales to compute the stochastic differential

$$d_s\left\{\left(u(s, X_s) - f(t, x) + \int_s^t (u'(r, X_r) - \mathscr{L}_r u(r, X_r))\,dr\right)\right.$$
$$\left.\times \exp\left\{\int_s^t (c(r, X_r) + h(r, X_r))\,dr\right\}\right\}.$$

Then find the integral $E^{t,x} \int_0^t$ of the resulting expression.

The existence of the fundamental solution to Eq. (3.2) is well-known (see [3]). For smooth f the representation formula in 3° follows from this fact and formula (3.7) for $h \equiv c \equiv 0$ and $u_0 = f$. The general result can be obtained now by passing to the limit. Q.E.D.

Formula (3.7) is usually referred to as Feynman–Kac formula. It provides a convenient representation for the backward diffusion $P^{t,x}$ which will play a central role in our further arguments. Note that 3° implies that $P^{t,x}$ has a transition density which can be identified with $p_{t,s}(x, y)$.

Remark 2. Let us assume that in addition to A_1–A_2 the assumption A_4 also holds. Then we can find a Lipschitz continuous in x and continuous in t $d \times d$-matrix $\sigma(t, x)$ so that $\sigma\sigma^T = a$. Suppose that $B = \{B_t, t \in [0, T]\}$ is a d-dimensional Wiener process independent of the white noise W. Let us consider the backward stochastic differential equation

$$\begin{cases} -dX_s^{t,x} = \sigma(s, X_s^{t,x})\, d\overleftarrow{B}_s + b(s, X_s^{t,x})\, ds \\ X_t^{t,x} = x. \end{cases} \tag{3.8}$$

It is well known that under assumptions A_1, A_2, A_4 this equation has a unique strong solution and for any $\Gamma \subset \mathscr{C}_t$, $P(X^{t,x} \in \Gamma) = P^{t,x}(\Gamma)$. Thus if A_1–A_4 holds, one can rewrite the Feynman–Kac formula (3.7) in the form

$$u^h(t, x) = E\left(u_0(X_0^{t,x}) \exp\left(\int_0^t [c(s, X_s^{t,x}) + h(s, X_s^{t,x})]\, ds \right)\right).$$

Proposition 3.5 (2°) implies the uniqueness of a Q-soft solution to Eq. (3.1) under hypotheses A_1–A_3. However in order to establish the existence of such a solution we have to introduce some suitable conditions on the operator Q.

For this purpose, let us consider the following hypotheses:

B_1: $Qe_k = \rho_k^2 e_k$, where $\{e_k, k \geqslant 1\}$ is a complete orthonormal system formed by bounded and Hölder continuous functions (uniformly in compact sets) e_k, and ρ_k is a nonincreasing sequence of positive real numbers such that $\rho_1 \leqslant 1$.

B_2: There exists $\lambda > 1$ such that for each $K > 0$ and $t \in (0, T]$ we have

$$\sup_{|x| \leqslant K} \sum_{N=0}^{\infty} \lambda^N \Lambda_t^N(x) < \infty,$$

where

$$\Lambda_t^N(x) := \int_{\{0 < s_1 < \cdots < s_N < t\}} \|(Q \otimes \cdots \otimes Q)(p_{t,s_N}(z, x_N) \times p_{s_N, s_{N-1}}(z_N, z_{N-1}) \cdots p_{s_2, s_1}(z_2, z_1))\|^2_{L^2((\mathbb{R}^d)^N)}\, ds_1 \cdots ds_N. \tag{3.9}$$

In (3.9) the notation $(Q\otimes\cdots\otimes Q)(p_{t,s_N}(x,z_N)\,p_{s_N,s_{N-1}}(z_N,z_{N-1}))$ means that given x we apply the direct product of N operators Q to the function

$$(z'_1,...,z'_N)\mapsto p_{t,s_N}(x,z'_N)\,p_{s_N,s_{N-1}}(z'_N,z'_{N-1})\cdots p_{s_2,s_1}(z'_2,z'_1)$$

and compute it at point $(z_1,...,z_N)$.

Remark 3. In dimension one condition B_2 holds if Q is the identity. In fact, we have

$$\begin{aligned}\Lambda^N_t(x)=\int_{\{0<s_1<\cdots<s_N<t\}}\int_{(\mathbb{R}^d)^N} & p_{t,s_N}(x,z_N)^2\\ &\times p_{s_N,s_{N-1}}(z_N,z_{N-1})^2\cdots p_{s_2,s_1}(z_2,z_1)^2\,dz_1\cdots dz_N\,ds_1\cdots ds_N,\end{aligned}$$

and using the inequality $p_{t,s}(x,y)^2\leqslant(c/\sqrt{t-s})\,p_{t,s}(x,y)$ (see [3]) we obtain

$$\begin{aligned}\Lambda^N_t(x)\leqslant\int_{\{0<s_1<\cdots<s_N<t\}} & c^N[(t-s_N)\\ &\times(s_N-s_{N-1})\cdots(s_2-s_1)]^{-1/2}\,ds_1\cdots ds_N=c^N\frac{\pi^{N/2}}{\Gamma(N/2+2)}\,t^{N/2+1}.\end{aligned}$$

Remark 4. Suppose that Q is a Hilbert–Schmidt operator whose kernel $Q(x,y)$ verifies

$$\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}Q(z,z_1)^2\,dz_1<\infty.$$

Then condition B_2 holds. Indeed, using Cauchy–Schwartz inequality yields

$$\begin{aligned}\Lambda^N_t(x)&=\int_{\{0<s_1<\cdots<s_N<t\}}\int_{(\mathbb{R}^d)^N}\bigg(\int_{(\mathbb{R}^d)^N}p_{t,s_N}(x,z_N)\\ &\quad\times p_{s_N,s_{N-1}}(z_N,z_{N-1})\cdots p_{s_2,s_1}(z_2,z_1)\,Q(z_N,z'_N)\,Q(z_{N-1},z'_{N-1})\\ &\quad\times\cdots\times Q(z_1,z'_1)\,dz_1\cdots dz_N\bigg)^2\,dz'_1\cdots dz'_N\,ds_1\cdots ds_N\\ &\leqslant\int_{\{0<s_1<\cdots<s_N<t\}}\int_{(\mathbb{R}^d)^{2N}}p_{t,s_N}(x,z_N)\\ &\quad\times p_{s_N,s_{N-1}}(z_N,z_{N-1})\cdots p_{s_2,s_1}(z_2,z_1)\,Q(z_N,z'_N)^2\\ &\quad\times\cdots\times Q(z_1,z'_1)^2\,dz_1\,dz'_1\cdots dz_N\,dz'_N\,ds_1\cdots ds_N\\ &\leqslant\frac{t^N}{N!}\left(\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}Q(z,z_1)^2\,dz_1\right)^N.\end{aligned}$$

Notice that

$$\sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} Q(z, z_1)^2 \, dz_1 = \sup_{z \in \mathbb{R}^d} \sum_{k=1}^{\infty} \rho_k^4 e_k^2(z) < \infty$$

provided the functions e_k verify

$$\sum_{k=1}^{\infty} \rho_k^4 \|e_k\|_\infty^2 < \infty. \tag{3.10}$$

Remark 5. In particular, for $Q = \prod_{i=1}^d (-\partial^2/\partial x_i^2 + x_i^2 + 1)^{-\alpha}$ with $\alpha > \frac{1}{2}$, condition (3.10), and so hypothesis B_2, is satisfied. Indeed, with the notation of Remark 1 we have

$$\rho_k = \prod_{j=1}^d (2\gamma(k)_j + 2)^{-\alpha/2},$$

which implies, taking into account the estimates given in [5], that

$$\begin{aligned} \sum_{k=1}^{\infty} \rho_k^4 \|e_k\|_\infty^2 &\leqslant C^2 \sum_{k=1}^{\infty} k^{-d/6} \prod_{j=1}^d (2\gamma(k)_j + 2)^{-2\alpha} \\ &\leqslant C^2 \left(\sum_{n=0}^{\infty} (2n+2)^{-2\alpha} \right)^d < \infty. \end{aligned}$$

Let us denote by $(\bar{\mathscr{X}}, \bar{\mathscr{C}}, \bar{\mathscr{C}}_s)$ another copy of the space $(\mathscr{X}, \mathscr{C}, \mathscr{C}_s)$. The canonical process on this space will be denoted $\bar{X}_s$. Let $\bar{P}^{t,y}$ be a solution to the martingale problem for $\mathscr{L}_s$ starting from (t, y) on the probability space $(\bar{\mathscr{X}}, \bar{\mathscr{C}}, \bar{\mathscr{C}}_s)$ and $P^{t,x,y}$ be a probability measure on the product space $(\mathscr{X} \times \bar{\mathscr{X}}, \mathscr{C} \times \bar{\mathscr{C}})$ given by $P^{t,x,y} := P^{t,x} \times \bar{P}^{t,y}$. The expectation operators relative to the measures $\bar{P}^{t,y}$ and $P^{t,x,y}$ will be denoted $\bar{E}^{t,y}$ and $E^{t,x,y}$ respectively.

LEMMA 3.6. *Let Q be an operator on $L^2(\mathbb{R}^d)$ satisfying the above conditions B_1 and B_2. Then given $(t, x, y) \in [0, T] \times R^d \times R^d$, the sequence*

$$\sum_{k=1}^{M} \rho_k^2 \int_0^t e_k(X_s) \, e_k(\bar{X}_s) \, ds$$

converges in $L^2(\mathscr{X} \times \mathscr{X}, P^{t,x,y})$, as M tends to infinity, to a random variable which will be denoted by

$$\int_0^t Q(X_s, \bar{X}_s) \, ds.$$

Moreover this random variable verifies

$$\sup_{t \leqslant T,\, |x| \leqslant K} E^{t,x,x}\left(\exp\left(\int_0^t Q(X_s, \bar{X}_s)\, ds\right)\right) < \infty, \tag{3.11}$$

for any $K > 0$, and for some $p > 1$ the sequence

$$\left\{\exp\left(\sum_{k=1}^{M} \rho_k^2 \int_0^t e_k(X_s)\, e_k(\bar{X}_s)\, ds\right),\ M \geqslant 0\right\}$$

is bounded in $L^p(\mathscr{X} \times \mathscr{X}, P^{t,x,x})$ uniformly in (t, x) on compact sets.

Proof. Let us first check the convergence in $L^2(\Omega)$. Fix two positive integers $M > N$. We have for each $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&E^{t,x,y}\left(\left|\sum_{k=N}^{M} \rho_k^2 \int_0^t e_k(X_s)\, e_k(\bar{X}_s)\, ds\right|^2\right) \\
&\quad = \sum_{j,k=N}^{M} \rho_k^2 \rho_j^2 \int_0^t \int_0^t E^{t,x,y}[e_k(X_s)\, e_k(\bar{X}_s)\, e_j(X_{s'})\, e_j(\bar{X}_{s'})]\, ds\, ds' \\
&\quad = 2 \sum_{j,k=N}^{M} \rho_k^2 \rho_j^2 \int_{\{0 < s' < s < t'\}} \int_{(\mathbb{R}^d)^4} p_{t,s}(x, z)\, e_k(z)\, p_{s,s'}(z, z')\, e_j(z') \\
&\qquad \times p_{t,s}(y, w)\, e_k(w)\, p_{s,s'}(w, w')\, e_j(w')\, dz\, dz'\, dw\, dw'\, ds\, ds' \\
&\quad \leqslant 2 \left(\int_{\{0 < s' < s < t\}} \sum_{j,k=N}^{M} \rho_k^2 \rho_j^2 \left|\int_{(\mathbb{R}^d)^2} p_{t,s}(x, z)\, p_{s,s'}(z, z')\right.\right. \\
&\qquad \left.\left. \times e_k(z)\, e_j(z')\, dz\, dz'\right|^2 ds\, ds'\right)^{1/2} \\
&\qquad \times \left(\int_{\{0 < s' < s < t\}} \sum_{j,k=N}^{M} \rho_k^2 \rho_j^2 \left|\int_{(\mathbb{R}^d)^2} p_{t,s}(y, z)\, p_{s,s'}(z, z')\right.\right. \\
&\qquad \left.\left. \times e_k(z)\, e_j(z')\, dz\, dz'\right|^2 ds\, ds'\right)^{1/2},
\end{aligned}$$

and this converges to zero as N and M tend to infinity due to condition B_2. In order to show the last part of the lemma concerning the exponential integrability, it suffices to check that

$$\sup_{M} \sup_{t \leqslant T,\, |x| \leqslant K} E^{t,x,x}\left(\lambda \exp\left(\sum_{k=1}^{M} \rho_k^2 \int_0^t e_k(X_s)\, e_k(\bar{X}_s)\, ds\right)\right) < \infty,$$

for each $K>0$, where λ is the constant appearing in condition B_2. We have

$$E^{t,x,x}\left(\exp\left(\sum_{k=1}^{M}\rho_k^2\lambda\int_0^t e_k(X_s)\,e_k(\bar{X}_s)\,ds\right)\right)$$

$$=\sum_{N=0}^{\infty}\lambda^N\int_{\{0<s_1<\cdots<s_N<t\}}E\left(\left(\sum_{k=1}^{M}\rho_k^2 e_k(X_{s_1})\,e_k(\bar{X}_{s_1})\right)\right.$$

$$\left.\times\cdots\times\left(\sum_{k=1}^{M}\rho_k^2 e_k(X_{s_N})\,e_k(\bar{X}_{s_N})\right)\right)ds_1\cdots ds_N$$

$$=\sum_{N=0}^{\infty}\lambda^N\int_{\{0<s_1<\cdots<s_N<t\}}\sum_{k_1,\ldots,k_N=1}^{M}\rho_{k_1}^2\cdots\rho_{k_N}^2$$

$$\times\left(E^{t,x,x}[e_{k_1}(X_{s_1})\cdots e_{k_N}(X_{s_N})]\right)^2 ds_1\cdots ds_N$$

$$=\sum_{N=0}^{\infty}\lambda^N\int_{\{0<s_1<\cdots<s_N<t\}}\sum_{k_1,\ldots,k_N=1}^{M}\rho_{k_1}^2\cdots\rho_{k_N}^2$$

$$\times\left(\int_{(\mathbb{R}^d)^N}e_{k_1}(y_1)\cdots e_{k_N}(y_N)\,p_{t,s_N}(x,y_N)\,p_{s_N,s_{N-1}}(y_N,y_{N-1})\right.$$

$$\left.\times\cdots\times p_{s_2,s_1}(y_2,y_1)\,dy_1\cdots dy_N\right)^2 ds_1\cdots ds_N$$

$$\leqslant\sum_{N=0}^{\infty}\lambda^N A_T^N(x).\qquad\text{Q.E.D.}$$

THEOREM 3.7. *Let Q be an operator on $L^2(\mathbb{R}^d)$ satisfying conditions B_1 and B_2. Then under assumptions A_1–A_3 there exists a unique Q-soft solution to Eq.* (3.1). *This solution is given by*

$$u(t,x)=\sum_{\alpha\in\mathcal{J}}\frac{1}{\sqrt{\alpha!}}E^{t,x}\left(u_0(X_0)\,e^{\int_0^t c(x,X_s)\,ds}\right.$$

$$\left.\times\prod_{j,k=1}^{\infty}\left(\int_0^t m_j(s)\,e_k(X_s)\,ds\right)^{\alpha_{jk}}\right)\xi_\alpha,\tag{3.12}$$

and for every $K<\infty$, $\sup_{|x|\leqslant K,\,t\leqslant T}\|u(t,x)\|_Q^2<\infty$.

We remark that as formula (3.11) shows, the Q-soft solution u does not depend on the choice of the operator Q. The latter is needed so as to describe the stochastic support $(L_Q^2(\Omega))$ of the solution.

Let $\mathcal{Z}$ be the set of real-valued double sequences $z=(z_{jk},\ j,k\geqslant 1)$ such that only a finite number of entries z_{jk} is not zero. For $z\in\mathcal{Z}$ set

$h_z=\sum_{j,k=1}^{\infty} z_{jk}(m_j\otimes e_k)$. Then, for any generalized random field $u=\{u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\}$ such that $u(t,x)\in L^2_Q(\Omega)$ we have

$$u^{h_z}(t,x)=E(u(t,x)\,q(h_z))=\sum_{\alpha\in\mathcal{J}}\lambda_\alpha z^\alpha, \tag{3.13}$$

where $z^\alpha=\prod_{j,k=1}^{\infty} z_{jk}^{\alpha_{jk}}$, and

$$\lambda_\alpha=\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial z^\alpha}(u^{h_z}(t,x))|_{z=0}.$$

We have that $E(u(t,x)\,\xi_\alpha)=\sqrt{\alpha!}\,\lambda_\alpha$, and the Fourier coefficients of $u(t,x)$ are determined by the constants λ_α. After these preliminaries we can proceed now with the proof of Theorem 3.7:

Proof of Theorem 3.7. Let u be a Q-soft solution of (3.12). Fix z in $\mathcal{Z}$. The deterministic function $u^{h_z}(t,x)$ is a generalized solution of

$$\begin{cases}\dfrac{\partial u^{h_z}}{\partial t}=\mathcal{L}u^{h_z}+u^{h_z}h_z\\ u^{h_z}(0,x)=u_0(x).\end{cases} \tag{3.14}$$

Then, the uniqueness part in the theorem is a consequence of the uniqueness of solution to this equation for any bounded and Hölder continuous function h_z (see Proposition 3.5). On the other hand, taking into account Feynman–Kac's representation we deduce

$$u^{h_z}(t,x)=E^{t,s}\left(u_0(X_0)\,e^{\int_0^t c(s,X_s)\,ds}\exp\left(\sum_{j,k=1}^{\infty} z_{jk}\int_0^t m_j(s)\,e_k(X_s)\,ds\right)\right).$$

Set

$$R=u_0(X_0)\,e^{\int_0^t c(s,X_s)\,ds}.$$

With this notation we can write

$$\begin{aligned}u^{h_z}(t,x)&=\sum_{N=0}^{\infty}\frac{1}{N!}E^{t,x}\left(R\left(\sum_{j,k=1}^{\infty} z_{jk}\int_0^t m_j(s)\,e_k(X_s)\,ds\right)^N\right)\\ &=\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{j_1,\dots,j_N=1}^{\infty}\sum_{k_1,\dots,k_N=1}^{\infty} z_{j_1k_1}\cdots z_{j_Nk_N}\\ &\quad\times E^{t,x}\left(R\prod_{l=1}^{N}\int_0^t m_{j_l}(s)\,e_{k_l}(X_s)\,ds\right)\\ &=\sum_{\alpha\in\mathcal{J}}\frac{1}{\alpha!}z^\alpha E^{t,x}\left(R\prod_{j,k=1}^{\infty}\left(\int_0^t m_j(s)\,e_k(X_s)\,ds\right)^{\alpha_{jk}}\right).\end{aligned}$$

Consequently,

$$\lambda_\alpha = \frac{1}{\alpha!} E^{t,x}\left(R \prod_{j,k=1}^{\infty} \left(\int_0^t m_j(s)\, e_k(X_s)\, ds\right)^{\alpha_{jk}}\right).$$

This implies that the unique Q-soft solution to equation (3.1) is the generalized random field given by (3.12). In order to complete the proof of the proposition we have to check that u is adapted and $\int_0^T \int_{\{|x| \leqslant K\}} \|u(t,x)\|_Q^2\, dx\, dt < \infty$.

Let us compute

$$\begin{aligned}\|u(t,x)\|_Q^2 &= \sum_{\alpha \in \mathscr{J}} (\rho^\alpha)^2 \,(E(u(t,x)\,\xi_\alpha))^2 \\ &= \lim_{K,L\to\infty} \sum_{\alpha\in\mathscr{J},\, j_\alpha \leqslant L,\, k_\alpha \leqslant K} \frac{1}{\alpha!} E^{t,x,x} \\ &\quad\times\left(R\bar{R} \prod_{j,k=1}^{\infty}\left(\int_0^t m_j(s)\,\rho_k e_k(X_s)\, ds\right)^{\alpha_{jk}}\right. \\ &\quad\times \left.\prod_{j,k=1}^{\infty}\left(\int_0^t m_j(s)\,\rho_k e_k(\bar{X}_s)\, ds\right)^{\alpha_{jk}}\right),\end{aligned}$$

where

$$\bar{R} = u_0(\bar{X}_0)\, e^{\int_0^t c(s,\bar{X}_s)\, ds},$$

$$j_\alpha = \min\{j\colon \alpha_{j'k} = 0,\ \forall j' > j,\ \forall k\}$$

and

$$k_\alpha = \min\{k\colon \alpha_{jk'} = 0,\ \forall j,\ \forall k' > k\}.$$

Now making use of the elementary formula $\sum_{|\alpha|=N} a_1^{\alpha_1} \cdots a_K^{\alpha_K}/\alpha_1! \cdots \alpha_K! = 1/N!\, (\sum_{i=1}^K a_i)^N$, we arrived at

$$\begin{aligned}\|u(t,x)\|_Q^2 &= \lim_{K,L\to\infty} \sum_{N=0}^{\infty} \frac{1}{N!} E^{t,x,x}\left(R\bar{R}\left(\sum_{j=1}^{L}\sum_{k=1}^{K}\left(\int_0^t m_j(s)\,\rho_k e_k(X_s)\, ds\right)\right.\right. \\ &\quad\times\left.\left.\left(\int_0^t m_j(s)\,\rho_k e_k(\bar{X}_s)\, ds\right)\right)^N\right) \\ &= \lim_{K,L\to\infty} E^{t,x,x}\left(R\bar{R}\exp\left(\sum_{j=1}^{L}\sum_{k=1}^{K}\left(\int_0^t m_j(s)\,\rho_k e_k(X_s)\, ds\right)\right.\right. \\ &\quad\times\left.\left.\left(\int_0^t m_j(s)\,\rho_k e_k(\bar{X}_s)\, ds\right)\right)\right) \\ &= \lim_{K\to\infty} E^{t,x,x}\left(R\bar{R}\exp\left(\int_0^t \sum_{k=1}^{K} \rho_k^2 e_k(X_s)\, e_k(\bar{X}_s)\, ds\right)\right). \qquad (3.15)\end{aligned}$$

Now using Lemma 3.5 we deduce

$$\|u(t,x)\|_Q^2 = E^{t,x,x}\left(R\bar{R}\exp\left(\int_0^t Q(X_s,\bar{X}_s)\,ds\right)\right).$$

As a consequence we have

$$\sup_{|x|\leqslant K,\, t\leqslant T} \|u(t,x)\|_Q^2 < \infty.$$

In order to show the adaptability of the generalized random field $u(t,x)$, it suffices to show that

$$E(u(t,x)\,q(h)) = E(u(t,x)\,E(q(h)\mid \mathscr{F}_t)), \tag{3.16}$$

for any $h\in L^2([0,T]\times\mathbb{R}^d)$. Notice that $E(q(g)\mid\mathscr{F}_t) = q(h\mathbf{1}_{[0,t]})$. Therefore, (3.16) is equivalent to seeing that $u^h(t,x) = u^{h\mathbf{1}_{[0,t]}}(t,x)$, and this is clear because both members of this equality are solutions of (3.2). Q.E.D

Now we will derive a stochastic Feynman–Kac formula for $u(t,x)$, the Q-soft solution to Eq. (3.1).

Denote $w_t^k = W(1_{[0,t]}\otimes e_k)$. Obviously w_t^t, $k=1,2,...$, is a system of independent standard Brownian motions.

Given $X\in\mathscr{X}$, define

$$:\exp\left\{\int_0^t W_N(ds,X_s)\right\}: := \exp\left\{\sum_{i=1}^N \int_0^t e_i(X_s)\,dw_s^i - \tfrac{1}{2}\sum_{i=1}^N \int_0^t e_i^2(X_s)\,ds\right\}$$

and

$$u_N(t,x) := E^{t,x}\left[R :\exp\left\{\int_0^t W_N(ds,X_s)\right\}:\right].$$

Recall that $R = u_0(X_0)\exp\{\int_0^t c(s,X_s)\,ds\}$.

Lemma 3.8. *Let Q be an operator on $L^2(\mathbb{R}^d)$ satisfying conditions B_1, B_2. Then under assumptions A_1–A_3 for every $(t,x)\in[0,T]\times R^d$, the sequence $u_N(t,x)$ converges strongly in $L_Q^2(\Omega)$ to a distribution denoted below by*

$$E^{tx}\left[u_0(X_0)\exp\left\{\int_0^t c(s,X_s)\,ds\right\} :\exp\left\{\int_0^t W(ds,X_s)\right\}:\right].$$

Proof. To prove that u_N, $N=1,2,...$ is a Cauchy sequence in $L_Q^2(\Omega)$, we need a convenient representation for the expression

$$E[\xi_\alpha(u_{N+p}-u_N)]$$
$$=E\left[\xi_\alpha E^{t,x}R\left(:\exp\left\{\int_0^t W_{N+p}(ds,X_s)\right\}: - :\exp\left\{\int_0^t W_N(ds,X_s)\right\}:\right)\right].$$

Using the well-known representation for the Wick polynomials

$$\xi_\alpha=\frac{1}{\sqrt{\alpha!}}\frac{\partial^{|\alpha|}}{\partial z^\alpha}\exp\left\{\sum_{k=1}^\infty\int_0^T m^k(z,s)\,dw_s^k-\frac{1}{2}\int_0^T\left(\sum_{k=1}^\infty m^k(z,s)\right)^2 ds\right\}_{|z=0},$$

where $m^k(z,s):=\sum_{j=1}^\infty z_{jk}m_j(s)$ and $z\in\mathscr{Z}$ (cf. (3.12)) and the fact that $u_N(t,x)$ is an adapted random field, it is easy to show that

$$Eu_N\xi_\alpha=\frac{1}{\sqrt{\alpha!}}\frac{\partial^{|\alpha|}}{\partial z^\alpha}EE^{t,x}\left[R\exp\left\{\sum_{k=1}^N\int_0^t(e_K(X_s)+m^k(z,s))\,dw_s^k\right.\right.$$
$$\left.\left.-\frac{1}{2}\sum_{k=1}^N\int_0^t(e_k(X_s)+m^k(z,s))^2\,ds+\sum_{k=1}^N\int_0^t e_k(X_s)\,m^k(z,s)\,ds\right\}\right]_{|z=0}$$
$$=\frac{1}{\sqrt{\alpha!}}\frac{\partial^{|\alpha|}}{\partial z^\alpha}E^{t,x}\left[R\exp\left\{\sum_{k=1}^N\int_0^t e_k(X_s)\,m^k(z,s)\,ds\right\}\right]_{|z=0},$$

where the latter equality follows from the Girsanov theorem. Now using the above formula we arrive at

$$E[\xi_\alpha(u_{N+p}-u_N)]$$
$$=\frac{1}{\sqrt{\alpha!}}\frac{\partial^{|\alpha|}}{\partial z^\alpha}E^{t,s}\left[R\exp\left\{\sum_{k=1}^N\int_0^t e_k(X_s)\,m^k(z,s)\,ds\right\}\right.$$
$$\left.\times\left(\exp\left\{\sum_{k=N+1}^{N+p}\int_0^t e_k(X_s)\,m^k(z,s)\,ds\right\}-1\right)\right]_{|z=0}. \tag{3.17}$$

From (3.17) it follows that

$$\|u_{N+p}-u_N\|_Q^2=\sum_{\alpha\in J,\,|\alpha|\neq 0}\frac{\rho^{2\alpha}}{\alpha!}\left(\frac{\partial^{|\alpha|}}{\partial z^\alpha}E^{t,x}\left[R\exp\left\{\sum_{k=1}^N\int_0^t e_k(X_s)\,m^k(z,s)\,ds\right\}\right.\right.$$
$$\left.\left.\times\left(\exp\left\{\sum_{k=N+1}^{N+p}\int_0^t e_k(X_s)\,m^k(z,s)\,ds\right\}-1\right)\right]_{|z=0}\right)^2$$
$$=\sum_{\alpha\in J,\,|\alpha|\neq 0}\left[E^{t,x}\left(R\prod_{j=1}^\infty\prod_{k=1}^N\frac{\rho_k^{\alpha_{jk}}}{\sqrt{\alpha_{jk}!}}\left(\int_0^t e_k(X_s)\,m_j(s)\,ds\right)^{\alpha_{jk}}\right.\right.$$
$$\left.\left.\times\prod_{j=1}^\infty\prod_{k=N+1}^{N+p}\frac{\rho_k^{\alpha_{jk}}}{\sqrt{\alpha_{jk}!}}\left(\int_0^t e_k(X_s)\,m_j(s)\,ds\right)^{\alpha_{jk}}\right)\right]^2. \tag{3.18}$$

Let $\mathscr{J}_n^{n+p}$ be the set of multiindices $\alpha = (\alpha_{kj} \geqslant 0;\ k = n, n+1, ..., n+p,\ j = 1, 2, ...)$ such that $|\alpha| \neq 0$. With this notation, making use of (3.18), we can write

$$
\begin{aligned}
&\|u_{N+p} - u_N\|_Q^2 \\
&\quad = \sum_{\alpha \in \mathscr{J}_1^N} E^{t,x,x}\Bigg(R\bar{R} \prod_{j=1}^{\infty} \prod_{k=1}^{N} \frac{1}{\alpha_{jk}!}\bigg(\int_0^t \rho_k e_k(X_s)\, m_j(s)\, ds \\
&\qquad \times \int_0^t \rho_k e_k(\bar{X}_s)\, m_j(s)\, ds\bigg)^{\alpha_{jk}}\Bigg) \\
&\qquad \times \sum_{\alpha \in \mathscr{J}_{N+1}^{N+p}} E^{t,x,x} \prod_{j=1}^{\infty} \prod_{k=N+1}^{N+p} \frac{1}{\alpha_{jk}!}\bigg(\int_0^t \rho_k e_k(X_s)\, m_j(s)\, ds \\
&\qquad \times \int_0^t \rho_k e_k(\bar{X}_s)\, m_j(s)\, ds\bigg)^{\alpha_{jk}} \\
&\quad = E^{t,x,x}\Bigg[R\bar{R}\left(\exp\left\{\sum_{k=1}^{N} \int_0^t \rho_k^2 e_k(X_s)\, e_k(\bar{X}_s)\, ds\right\} - 1\right) \\
&\qquad \times \left(\exp\left\{\sum_{k=N+1}^{N+p} \int_0^t \rho_k^2 e_k(X_s)\, e_k(\bar{X}_s)\, ds\right\} - 1\right)\Bigg] \\
&\quad = E^{t,x,x}\Bigg[R\bar{R}\left(\exp\left\{\sum_{k=1}^{N+p} \int_0^t \rho_k^2 e_k(X_s)\, e_k(\bar{X}_s)\, ds\right\}\right. \\
&\qquad - \exp\left\{\sum_{k=1}^{N} \int_0^t \rho_k^2 e_k(X_s)\, e_k(\bar{X}_s)\, ds\right\} \\
&\qquad \left. - \exp\left\{\sum_{k=N+1}^{N+p} \int_0^t \rho_k e_k(X_s)\, e_k(\bar{X}_s)\, ds\right\} + 1\right)\Bigg].
\end{aligned}
$$

By Lemma 3.6, the right-hand side of the equality converges to zero as $N \to \infty$ and $p \to \infty$. Q.E.D.

PROPOSITION 3.9 (Feynman–Kac's Formula). *Let Q be an operator on $L^2(\mathbb{R}^d)$ satisfying conditions B_1, B_2. Assume also that A_1–A_3 hold. Let $u(t, x)$ be a Q-soft solution to Eq.* (3.1), *then for every $(t, x) \in [0, T] \times \mathbb{R}^d$*

$$
u(t, x) = E^{t,x}\left[u_0(X_0) \exp\left\{\int_0^t c(s, X_s)\, ds\right\} :\exp\left\{\int_0^t W(ds, X_s)\right\}:\right]. \quad (3.19)
$$

Proof. Making use of standard arguments (see e.g., [16, lemma 1.1.2] one can show that the set of random variables

$$q(h_z) = \exp\left\{\sum_{k=1}^{\infty} \int_0^T m_k(z, s)\, dw_s^k - \frac{1}{2}\int_0^T \left(\sum_{k=1}^{\infty} m^k(z, s)\right)^2 ds\right\}, \qquad z \in \mathscr{Z}$$

is total in $L^2_{Q^-}(\Omega)$. Hence it suffices to prove that

$$Eu(t, x)\, q(h_z) = Ev(t, x)\, q(h_z) \qquad \text{for all} \quad z \in \mathscr{Z},$$

where

$$v(t, x) := E^{t, x}\left[R{:}\exp\left\{\int_0^t W(ds, X_s)\right\}{:}\right].$$

From Lemma 3.8 we have

$$Ev(t, x)\, q(h_z) = \lim_{N \to \infty} Eu_N(t, x)\, q(h_z).$$

On the other hand repeating the arguments used in the proof of Lemma 3.6 one can show that

$$Eu_N(t, x)\, q(h_z) = E^{t, x}\left[R \exp\left\{\sum_{k=1}^{N} m^k(z, s)\, e_k(X_s)\right\}\right].$$

Since by the definition of $\mathscr{Z}$ only a finite number of the entries z_{jk} in the vector z is not zero, it is readily checked that

$$\lim_{N \to \infty} Eu_N(t, x)\, q(h_z) = E^{t, x}\left[R \exp\left\{\sum_{k=1}^{\infty} m^k(z, s)\, e_k(X_s)\right\}\right]. \tag{3.20}$$

Combining now (3.20) with (3.13) and (3.14) we arrive at

$$\lim_{n \to \infty} Eu_N(t, x)\, q(h_z) = Eu(t, x)\, q(h_z) \qquad \text{for all} \quad z \in \mathscr{Z}.$$

Q.E.D.

The following result is a straightforward corollary of Proposition 3.9.

PROPOSITION 3.10 (Maximum Principle). *If the assumptions of Proposition 3.9 hold and $u_0(x) \geqslant 0$ for all x, then for every $g \in L^2_{Q^-}(\Omega)$ so that $g \geqslant 0$ a.s.*

$$Eu(t, x)\, g \geqslant 0 \qquad \textit{for all} \quad t \in [0, T].$$

Remark 6. If in addition to the assumption A_1–A_3 we suppose that the coefficients b and $\sigma(t,x)=\sqrt{a(t,x)}$ are Lipschitz continuous in x on compact sets, then Formula (3.11) can be rewritten in the form

$$u(t,x)=\sum_{\alpha\in J}\frac{1}{\sqrt{\alpha!}}E\left(u_0(X_0^{t,x})\,e^{\int_0^t c(s,X_s^{t,x})\,ds}\right.$$
$$\left.\times\prod_{j,k=1}^{\infty}\left(\left(\int_0^t m_j(s)\,e_k(X_s^{t,x})\,ds\right)^{\alpha_{jk}}\right)\right)\xi_\alpha,$$

where $X_s^{t,x}$ is a strong solution of the equation (3.8). Similar changes can be made in the Feynman–Kac formula (3.19).

Remark 7. Assume that $d=1$ and take $Q=I$. Transformations similar to those used in (3.15) yield that for any test function $\varphi\in C^\infty(\mathbb{R}^d)$,

$$E\left(\left|\int_{R^d}u(t,x)\,\varphi(x)\,dx\right|^2\right)$$
$$=\sum_{N=0}^{\infty}\int_{\{0<s_1<\cdots<S_N<t\}}\sum_{k_1\cdots k_N=1}^{\infty}\left(\int_{R^d}E^{t,x}[e_{k_1}(X_{s_1})\cdots e_{k_N}(X_{s_N})\right.$$
$$\left.\times e^{\int_0^t c(X_n)\,dn}u_0(X_0)]\,\varphi(x)\,dx\right)^2 ds_1\cdots ds_N$$
$$\leqslant C\sum_{N=0}^{\infty}\int_{\{0<s_1<\cdots<s_N<t\}}\int_{(\mathbb{R}^d)^N}\left(\int_{\mathbb{R}^d}\varphi(x)\,p_{t,s_N}(x,y_N)\,dx\right)^2$$
$$\times p_{s_N,s_{N-1}}(y_N,y_{N-1})^2\cdots p_{s_2-s_1}(y_2,y_1)^2$$
$$\times\left(\int_{\mathbb{R}^d}u_0(y_0)\,p_{s_1,0}(y_1,y_0)\,dy_0\right)^2 dy_1\cdots dy_N\,ds_1\cdots ds_N.$$

Using the inequality $p_t^2(x,y)\leqslant c(1/\sqrt{t})\,p_t(x,y)$, and denoting by $\mathscr{D}$ a closed ball containing the support of φ, we get for any $\varepsilon>0$

$$E\left(\left|\int_{R^d}u(t,x)\,\varphi(x)\,dx\right|^2\right)\leqslant c\,\|u_0\|_\infty^2\,\|\varphi\|_{H^{-1/2+\varepsilon}(\mathscr{D})}^2.$$

Consequently, we have that $u(t)\in L^2(\Omega;H^{1/2-\varepsilon}(\mathscr{D}))$.

Remark 8. If we have $\sum_{k=1}^{\infty}\rho_k^2\,\|e_k\|_\infty^2<\infty$, then B_2 holds (see Remark 2 above) and, moreover, the kernel $Q(x,y)$ of the operator Q is bounded. In general, in dimension $d\geqslant 2$ conditions B_1 and B_2 hold provided Q is a Hilbert–Schmidt operator with bounded Hölder continuous eigenfunctions e_k such that $\sum_{k=1}^{\infty}\rho_k^4\,\|e_k\|_\infty^2<\infty$.

Notice that when $d>1$ we have, from the preceding computations, that $E\,\|u(t)\|_{H^{-k}(\mathscr{D})}^2=\infty$ for any k, and any bounded domain $\mathscr{D}$.

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